2.1

Angles and Coordinate Lines

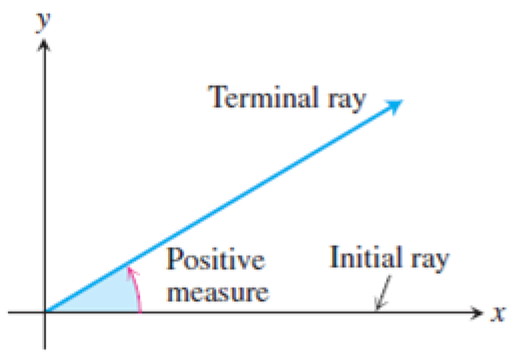
Learning objectives:

- To understand the plane angle and measures of an angle
- To convert radians to degrees and degrees to radians
- To understand the concept of arc length and the area of a sector and

  AND
- To practice problems related to the above concepts

Plane Angle:

Angles in the plane can be generated by rotating a ray about its endpoint. The starting position of the ray is called the initial ray (initial side) of the angle, the final position is called the terminal ray (terminal side) of the angle, and the point at which the initial and terminal rays meet is called the vertex of the angle.



An angle in the xy-plane is said to be in ***standard position*** if its vertex lies at the origin and its initial ray lies along the positive $x-axis$. Angles measured counterclockwise from the positive x −axis are assigned **positive measures**; angles measured clockwise are assigned **negative measures.**

Measures of Angles

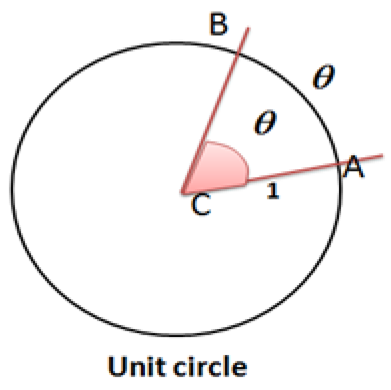
There are **two standard** measures for describing the size of an angle: **degree measure** and **radian measure.**

In degree measure, **one degree (written 1^o) is the measure of an angle generated by $\frac{1}{360}$ of one revolution.** Thus, there are 360^o in an angle of one revolution, $180°$ degrees in an angle of one-half revolution, $90°$ in an angle of one-quarter revolution (a right angle), and so forth.

One degree is divided into sixty equal parts, called **minutes,** and one minute is divided into sixty equal parts, called **seconds.** Thus, one minute (written $1'$) is $1/60$ of a degree, and one second (written $1''$) is $1/60$ of a minute.

In radian measure, angles are measured by the length of the arc that the angle subtends on a circle of radius 1 with its vertex at the center (unit circle).

Let ACB be a central angle in a unit circle.



The radian measure θ of angle ACB is defined to be the length of the circular arc AB.

Since the circumference of the circle is 2π and one complete revolution of a circle is $360°$, the relation between radians and degrees is given by the equation

$$\pi \text{ radians} = 180^\circ$$

So, $1 \; rad = \left(\frac{180}{\pi}\right)^\circ \doteq 57^\circ (57^\circ.17' \;\; approx.)$

$$1^\circ = \frac{\pi}{180} \text{ rad} \doteq 0.02 \text{ rad}$$

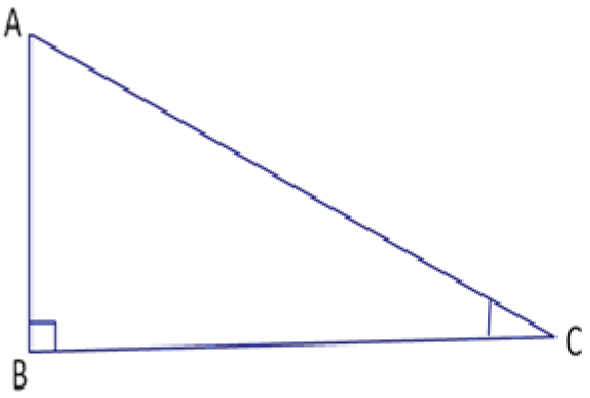
Example1:

Convert 45° to radians: $45\frac{\pi}{180} = \frac{\pi}{4}$ rad

Convert $\frac{\pi}{6}$ rad to degrees: $\frac{\pi}{6}\frac{180}{\pi} = 30°$

P1:

In a Right angled triangle ABC



If $\boldsymbol{a} = \mathbf{4}, \boldsymbol{b} = \mathbf{5}$ then the value of $\mathbf{10}\left[\frac{\sin A+\cos C}{\tan A+\cot C}\right] + \mathbf{18}\left[\frac{\csc A+\sec C}{\cot A+\tan C}\right] =$

A. 30
B. 36
C. 25
D. 17

Answer: B

Solution:

By hypothesis, we have $a = 4, b = 5$

By Pythagoras theorem, we have

$$(5)^2 = (4)^2 + AB^2$$

$$\Rightarrow 25 - 16 = AB^2$$

$$\Rightarrow AB^2 = 9$$

$$\Rightarrow AB = \sqrt{9} = 3 = c$$

Now,

$$10\left[\frac{\sin A+\cos C}{\tan A+\cot C}\right] + 18\left[\frac{\csc A+\sec C}{\cot A+\tan C}\right]$$

$$= 10\left[\frac{\frac{4}{5}+\frac{4}{5}}{\frac{4}{3}+\frac{4}{3}}\right] + 18\left[\frac{\frac{5}{4}+\frac{5}{4}}{\frac{3}{4}+\frac{3}{4}}\right]$$

$$= 10\left[\frac{\frac{4}{5}}{\frac{4}{3}}\right] + 18\left[\frac{\frac{5}{4}}{\frac{3}{4}}\right]$$

$$= 10\left[\frac{3}{5}\right] + 18\left[\frac{5}{3}\right]$$

$$= 6 + 30 = 36$$

Hence $10\left[\frac{\sin A+\cos C}{\tan A+\cot C}\right] + 18\left[\frac{\csc A+\sec C}{\cot A+\tan C}\right] = 36$

P4:

If $acos^3\theta + 3a\ cos\theta sin^2\theta = m$ and $asin^3\theta + 3acos^2\theta sin\theta = n$, then $(m+n)^{\frac{2}{3}} + (m-n)^{\frac{2}{3}} =$

A. $2\sqrt{a^3}$

B. $3\sqrt[3]{a^2}$

C. $3\sqrt{a^3}$

D. $2\sqrt[3]{a^2}$

Periodicity

When an angle of measure x and an angle of measure $x + 2\pi$ are in standard position, their terminal rays coincide. The two angles therefore have the same trigonometric values. For example, $\cos(x+2\pi) = \cos x$.

Trigonometric functions whose values repeat at regular intervals are called periodic.

Periodic function and period:

Let $A \subseteq \boldsymbol{R}$. A function $f: A \longrightarrow \boldsymbol{R}$ is said to be a ***periodic function*** if there exists a positive number p such that $f(x+p) = f(x)$, for all $x \in A$. Then ***p is said to be a period of f***. If p is the least positive real number such that $f(x+p) = f(x)$, for all $x \in A$.Then p is called ***the period of f***.

If f is a periodic function with the period p. Then

i) The function cf is also a periodic function with the period p, where c is a real number.
ii) $f(x+np) = f(x), \forall\, x \in A$ and $n \in \boldsymbol{Z}$.
iii) The function $g: A \longrightarrow \boldsymbol{R}$ defined by $g(x) = f(ax+b) + c\,, x \in A$, where $a, b, c \in \boldsymbol{R}, a \neq 0$ is also a periodic function with the period $\frac{p}{|a|}$

Note:

Let $f: A \rightarrow \boldsymbol{R}$, $g: B \rightarrow \boldsymbol{R}$ be two periodic functions with the periods p and q respectively.

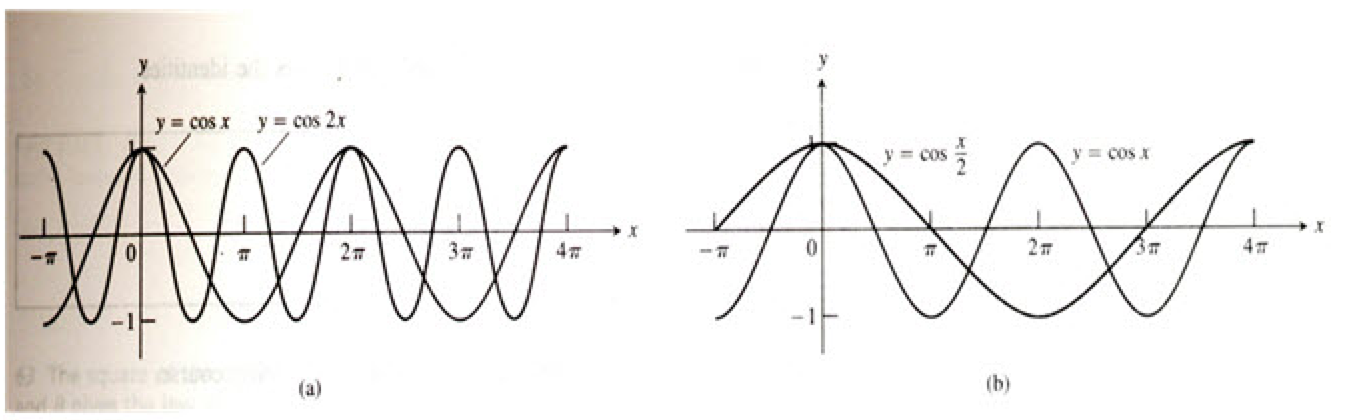
Then the functions $(f \mp g)$, (fg) and $\left(\frac{f}{g}\right)$ are all periodic functions with *a period* $\boldsymbol{r} = l.c.m.\{p, q\}$.In these cases the period is either $\boldsymbol{r}$ or a sub multiple of $\boldsymbol{r}$.

As seen from their graphs, the tangent and cotangent functions have the period$p = \pi$. The other four functions have the period2π.

$$\tan(x+\pi) = \tan x$$
$$\cot(x+\pi) = \cot x$$

$$\sin(x+2\pi) = \sin x$$
$$\cos(x+2\pi) = \cos x$$
$$\sec(x+2\pi) = \sec x$$
$$\csc(x+2\pi) = \csc x$$

The graphs of $y = \cos 2x$ *and* $y = \cos\left(\frac{x}{2}\right)$ plotted against the graph of $y = \cos x$ are shown below.



Multiplying x by a number greater than 1 speeds up a trigonometric function (increases the frequency) and shortens its period. Multiplying x by a positive number less than 1 slows a trigonometric function down and lengthens its period.

Even Versus Odd function:

The symmetries in the graphs of trigonometric functions reveal that the cosine and secant functions are even and the other four functions are odd.

Even functions	Odd functions
$\cos(-x) = \cos x$	$\sin(-x) = -\sin x$
$\sec(-x) = \sec x$	$\tan(-x) = -\tan x$
	$\csc(-x) = -\csc x$
	$\cot(-x) = -\cot x$

2.5

Reductions to Functions of Positive Acute Angles

Learning objectives:

- The values of Trigonometrical ratios of the angles $(90° - \theta), (90° + \theta), \ (180° - \theta), \ (180° + \theta),$ $(270° - \theta), \ (270° + \theta) \quad (360° - \theta), \ (360° + \theta).$

  And

- Co-terminal angles and Angles with a given trigonometric function value.

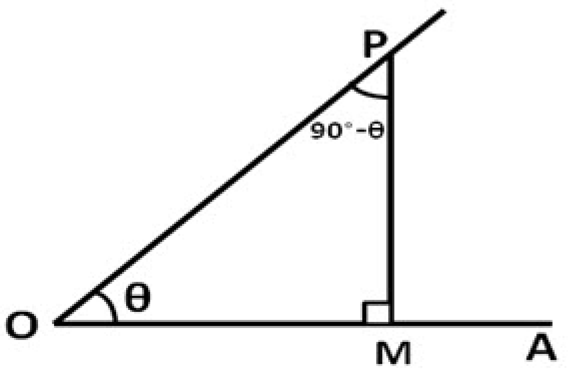
  And

- Solve the problems related to the above concepts.

Trigonometric Ratios of the Angle $(90° - \theta)$

Two angles are said to be ***complementary*** when their sum is equal to a right angle. Thus any angle θ and the angle $(90^o - \theta)$ are complementary.

Let the revolving line, starting from OA, trace out an acute angle AOP, equal to θ. From any point P on it draw PM perpendicular to OA. Clearly, the angles MOP and OPM are complementary and angle OPM $= 90^o - \theta$.



$$\sin(90^\circ - \theta) = \frac{OM}{OP} = \cos\theta, \qquad \cos(90^\circ - \theta) = \frac{MP}{OP} = \sin\theta$$

$$\tan(90^\circ - \theta) = \frac{OM}{MP} = \cot\theta, \qquad \cot(90^\circ - \theta) = \frac{PM}{OM} = \tan\theta$$

$$\csc(90^\circ - \theta) = \frac{OP}{OM} = \sec\theta, \qquad \sec(90^\circ - \theta) = \frac{OP}{PM} = \csc\theta$$

Hence we observe that

- The Sine of any angle is equal to the Cosine of its complement,
- The Tangent of any angle is equal to the Cotangent of its complement, and
- The Secant of any angle is equal to the Cosecant of its complement.

This shows how the names **co**sine, **co**tangent, and **co**secant are derived from sine, tangent and secant.

Example1: Evaluate $\frac{\sin 23^\circ}{\cos 67^\circ}, \frac{\csc 29^\circ}{\sec 61^\circ}$

Solution:

$$\frac{\sin 23^\circ}{\cos 67^\circ} = \frac{\sin(90-67)^\circ}{\cos 67^\circ} = \frac{\cos 67^\circ}{\cos 67^\circ} = 1$$

$$\frac{\csc 29^\circ}{\sec 61^\circ} = \frac{\csc(90-61)^\circ}{\sec 61^\circ} = \frac{\sec 61^\circ}{\sec 61^\circ} = 1$$

2.6

Trigonometric Functions of Two Angles

Learning Objectives:

- To derive the angle sum and the angle difference formulae of trigonometric functions

AND

- To solve problems related to above formulae

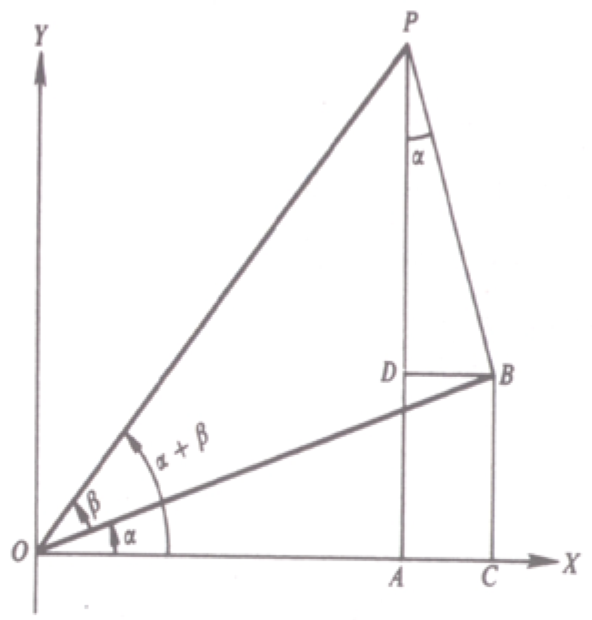
Angle Sum Formulas

We prove the following, the angle sum formulas.

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$



Let OX and OB be the initial and terminal rays of the angle α in the standard position. Let OP be the terminal ray of the angle $\alpha+\beta$. Let PA and PB be perpendiculars on OX and OB respectively. Draw $BC \perp OX$ and $DB \parallel OX$.

Now, angle OBD $= \alpha$

The angle APB = $90^\circ - \text{angle } DBP = \text{angle } OBD = \alpha$

$$\sin(\alpha+\beta)=\frac{PA}{OP}=\frac{PD+DA}{OP}=\frac{PD+BC}{OP}=\frac{PD}{OP}+\frac{BC}{OP}$$
$$=\frac{CB}{OB}\cdot\frac{OB}{OP}+\frac{PD}{PB}\cdot\frac{PB}{OP}$$
$$=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

$$\cos(\alpha+\beta)=\frac{OA}{OP}=\frac{OC-AC}{OP}=\frac{OC-DB}{OP}=\frac{OC}{OP}-\frac{DB}{OP}$$
$$=\frac{OC}{OB}\cdot\frac{OB}{OP}-\frac{DB}{PB}\cdot\frac{PB}{OP}$$
$$=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

$$\tan(\alpha+\beta)=\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}$$
$$=\frac{\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}-\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$
$$=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

2.7

Trigonometric Functions of Multiple Angles

Learning Objectives

- To derive double - angle and triple - angle formulae And
- To solve problems related to them

Double-Angle Formulas

We propose to find the trigonometrical functions of an angle 2θ in terms of those of the angle θ .

In the angle sum formulas

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$
$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$
$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

we substitute θ for both α and β. This gives three more identities:

$$\cos 2\theta=\cos^2\theta-\sin^2\theta$$
$$\sin 2\theta=2\sin\theta\cos\theta$$
$$\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}$$

These identities are known as ***double-angle*** formulas.

Example 1: Write the expression $2\sin 75^\circ\cos 75^\circ$ as a single function of an angle.

Solution: $2\sin 75^\circ\cos 75^\circ=\sin(2\times 75^\circ)=\sin 150^\circ$

Additional Double-angle Formulas

Additional formulas come from combining the equations

$$\cos^2\theta+\sin^2\theta=1$$

$$\cos^2\theta-\sin^2\theta=\cos 2\theta$$

We add and subtract the equations in turn to get the following formulas:

$$\cos^2\theta=\frac{1+\cos 2\theta}{2}$$
$$\sin^2\theta=\frac{1-\cos 2\theta}{2}$$

These formulas can also be written as

$$\cos 2\theta=2\cos^2\theta-1$$
$$\cos 2\theta=1-2\sin^2\theta$$

Example 2: Write the expression $1-2\sin^2 37^\circ$ as a single function of an angle.

Solution: $1-2\sin^2 37^\circ=\cos(2\times 37^\circ)=\cos 74^\circ$

Some Useful Identities

The following identities are useful in solving the problems. They are easily derived using the above formulas.

$$\sin 2\theta=\frac{2\tan\theta}{1+\tan^2\theta}$$
$$\cos 2\theta=\frac{1-\tan^2\theta}{1+\tan^2\theta}$$
$$\cot 2\theta=\frac{\cot^2\theta-1}{2\cot\theta}$$

2.8

Trigonometric Functions of Sub-multiple Angles

Learning objectives:

- We derive Half-angle formulae and one-third angle formulae of the trigonometric functions.
  And
- Solve the problems related to the above concepts.

Half-Angle Formulas

When θ is replaced by $\frac{\theta}{2}$, the resulting formulas are called **Half-angle** formulas. All the previous formulas will apply.

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \qquad ...(1)$$

$$\left.\begin{aligned}\cos\theta &= \cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}\\ &= 2\cos^2\frac{\theta}{2}-1\\ &= 1-2\sin^2\frac{\theta}{2}\end{aligned}\right] \qquad ...(2)$$

$$\tan\theta = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}} \qquad ...(3)$$

From (1), we also have

$$\sin\theta = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}}$$

We divide both numerator and denominator by $\cos^2\frac{\theta}{2}$. We obtain

$$\sin\theta = \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} \qquad ...(4)$$

From (2),

$$\cos\theta = \frac{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}}$$

Dividing both numerator and denominator by $\cos^2\frac{\theta}{2}$,

$$\cos\theta = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} \qquad ...(5)$$

Again from the formula (2), we can also obtain the formulas

$$\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}, \quad \cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}} \qquad ...(6)$$

$$\sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}, \quad \sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}} \qquad ...(7)$$

From this, we deduce that

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \qquad ...(8)$$

Example 1:

Write the expression $\sqrt{\frac{1-\cos 84^\circ}{2}}$ as a single function of an angle.

Solution: $\sqrt{\frac{1-\cos 84^\circ}{2}} = \sin\frac{84^\circ}{2} = \sin 42^\circ$

Example 2:

Find the value of $tan7\frac{1}{2}°$, $cot7\frac{1}{2}°$

Solution:

$$tan7\tfrac{1}{2}° = \frac{sin7\frac{1}{2}°}{cos7\frac{1}{2}°} = \frac{2sin^2 7\frac{1}{2}°}{2sin7\frac{1}{2}°\,cos7\frac{1}{2}°}$$

$$= \frac{1-cos15°}{sin15°} = \frac{1-\frac{\sqrt{3}+1}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}}$$

$$= \frac{2\sqrt{2}-\sqrt{3}-1}{\sqrt{3}-1} = \frac{(2\sqrt{2}-\sqrt{3}-1)(\sqrt{3}+1)}{3-1}$$

$$= \sqrt{6}-\sqrt{3}+\sqrt{2}-2 = (\sqrt{3}-\sqrt{2})(\sqrt{2}-1)$$

$$\therefore\ tan7\tfrac{1}{2}° = (\sqrt{3}-\sqrt{2})(\sqrt{2}-1)$$

$$\therefore\ cot7\tfrac{1}{2}° = \frac{1}{tan7\frac{1}{2}°} = (\sqrt{3}+\sqrt{2})(\sqrt{2}+1)$$

2.9

Inverse Trigonometric Functions

Learning Objectives:

- To define Inverse Trigonometric Functions
- To derive some formulae of Inverse Trigonometric Functions

  AND
- To solve related problems

Inverse Trigonometric Relations

The equation

$$x = sin\, y$$

defines a unique value of x for each given angle y. But when x is given, the equation may have no solution or many solutions. For example: if $= 2$, there is no solution, since the sine of an angle never exceeds 1. If $x = \frac{1}{2}$ there are many solutions $y = 30^o, 150^o, \ldots$.

The symbol $sin^{-1} x$ is used to denote the smallest angle, whether positive or negative, that has x for its sine.

$$y = \sin^{-1} x$$

The symbol $\sin^{-1} x$ is read in words as "inverse sine of x". This shall not be confused with $\frac{1}{\sin x}$ which would be written in the form $(\sin x)^{-1}$.

Thus $\mathbf{\sin^{-1} x}$ is an *angle*, and denotes the *smallest numerical* angle whose sine is $\boldsymbol{x}$.

An alternative notation for the inverse relation is

$$y = arc\ sin\, x$$

This equation is to be interpreted as stating that "y is an angle whose sine is x."

So, $\cos^{-1} x$ or $arc\ \cos x$ means the smallest numerical angle whose cosine is x. Similarly, the inverse of other trigonometric ratios are defined.